ECE 331 Introduction to Random Signal Analysis and Statistics

Homework #3 Suggested Solutions

Chapter 2: 12, 13, 14, 16, 17, 18, 20, 26, 27, 29, 34, 35, 36. Chapter 3: 3, 4, 5, 6, 10(a), 11(a), 19, 20(a), 21

1. (2-12, 4 points) $E[1/(X+1)] = \sum_{k=0}^{\infty} \frac{1}{k+1} \cdot \frac{l^k e^{-l}}{k!} = \frac{1}{l} \sum_{k=0}^{\infty} \frac{l^{k+1} e^{-l}}{(k+1)!} \overset{m=k+1}{=} \frac{1}{l} \sum_{m=1}^{\infty} \frac{l^m e^{-l}}{m!} = \frac{1-e^{-l}}{l}$
2. (2-13, 4 points) $E[X^2] = Var(X) + m^2 = 7 + 2^2 = 11$.
3. (2-14, 4 points) $G(z) = (1/6) + (1/6)z + (2/3)z^2$. $G'(z) = (1/6) + (4/3)z$, and $G^{(2)}(z) = 4/3$. However, $E[X] = G'(1) = (1/6) + (4/3) = 9/6 = 3/2$, $E[X(X-1)] = G^{(2)}(1) = 4/3 = E[X^2] - E[X]$. Hence $E[X^2] = E[X] + 4/3 = 3/2 + 4/3 = 17/6$, and $Var(X) = E[X^2] - (E[X])^2 = 17/6 - (3/2)^2 = 7/12$.
4. (2-16, 4 points) $G(z) = [(1-p)+pz]^n$. $E[Y] = G'(1) = n[(1-p)+pz]^{n-1}p\big|_{z=1} = np$. $G^{(2)}(z) = n(n-1)[(1-p)+pz]^{n-2}p^2$, and $E[Y(Y-1)] = G^{(2)}(1) = n(n-1)p^2$. Thus, $E[Y^2] = n(n-1)p^2 + np$, and $Var(Y) = E[Y^2] - (E[Y])^2 = (n(n-1)p^2 + np) - (np)^2 = np - np^2 = np(1-p)$.
5. (2-17, 4 points) $X \sim Poisson(l)$ Þ $G_X(z) = \exp(l(z-1))$. $Y = X_1 + \ldots + X_n$. $E[z^Y] = E[z^{X1+\ldots+Xn}] = E[z^{X1}] \ldots E[z^{Xn}] = [\exp(l(z-1))]^n = \exp(n\,l(z-1))$.
6. (2-18, 4 points) $\binom{n}{k} p^k (1-p)^{n-k}$
7. (2-20, 4 points) Since X and Y are independent r.v.s, $E[(X+Y)^3] = E[X^3] + 3E[X^2]E[Y] + 3E[X]E[Y^2] + E[Y^3]$. $G_X(z) = G_Y(z) = (1-p) + pz$. $E[X]=E[Y] = G'(1) = p$, $E[X(X-1)] = G^{(2)}(1) = 0$, *and* $E[X(X-1)(X-2)] = G^{(3)}(1) = 0$. Thus, $E[X^2] = E[Y^2] = p$, and $E[X^3] = E[Y^3] = 3E[X^2] - 2E[X] = 3p - 2p = p$. Hence, $E[(X+Y)^3] = p+3p^2+3p^2+p = 6p^2 + 2p$.
8. (2-26, 4 points) $P(Y<2\,|\,X=i) = P(Y=1\,|\,X=i) + P(Y=0\,|\,X=i) = np_i(1-p_i)^{n-1} + (1-p_i)^n$
9. (2-27, 4 points) $P(X>2\,|\,Y=j) = \sum_{k=3}^{\infty} \frac{l_j^k e^{-l_j}}{k!} = 1 - e^{-l_j}(1 + l_j + l_j^2/2)$
10. (2-29, 6 points) Assume that only 1 or 2 can be transmitted. Define a Bernoulli random variable T ~ Bernoulli(0.5) such that T = 1 or 2 with equal probability of 0.5. We then have $P(R=r|T=1) \sim Poisson(m)$, $P(R=r|T=2) \sim Poisson(n)$. To find $P(T=1|R=2) = P(R=2|T=1)P(T=1)/P(R=2)$. Use law of total probability,

$$P(R=2) = P(R=2\,|\,T=1)P(T=1) + P(R=2\,|\,T=2)P(T=2)$$
$$= \frac{m^2 e^{-m}}{2}(0.5) + \frac{n^2 e^{-n}}{2}(0.5) = \frac{m^2 e^{-m} + n^2 e^{-n}}{4}$$

Thus, $P(T=1\,|\,R=2) = \frac{m^2 e^{-m}}{4} \Big/ \frac{m^2 e^{-m} + n^2 e^{-n}}{4} = \frac{m^2 e^{-m}}{m^2 e^{-m} + n^2 e^{-n}} = \frac{1}{1+(n/m)^2 e^{m-n}}$

11. (2-34, 6 points) (i) Recall that $p_X(X=1) = 1/3$, and $p_X(X=2) = 2/3$. We have,

$$p_{Y|X}(Y=j \mid X=i) = \frac{p_{XY}(i,j)}{p_X(i)} = \begin{cases} 3^j e^{-3}/j! & i=1, j \ge 0 \\ 6^j e^{-6}/j! & i=2, j \ge 0 \end{cases}.$$ Thus,

$$E[Y \mid X=i] = \begin{cases} \sum_{j=0}^{\infty} j \cdot 3^j e^{-3}/j! & i=1, \\ \sum_{j=0}^{\infty} j \cdot 6^j e^{-6}/j! & i=2, \\ 0 & otherwise. \end{cases} = \begin{cases} 3 & i=1, \\ 6 & i=2, \\ 0 & otherwise. \end{cases}$$

(ii) From problem 2-11, $p_Y(j) = \dfrac{3^{j-1}e^{-3} + 4\cdot 6^{j-1}e^{-6}}{j!}; \quad j \ge 0.$ Hence,

$$E[Y] = \sum_{j=0}^{\infty} j p_Y(j) = \sum_{j=0}^{\infty} j \frac{3^{j-1}e^{-3} + 4\cdot 6^{j-1}e^{-6}}{j!} = 1 + 4 = 5.$$ Or alternatively,

$E[Y]=E[Y|X=1]P(X=1)+E[Y|X=2]P(X=2) = 3(1/3) + 6(2/3) = 1 + 4 = 5.$

(iii) $E[X|Y=j] = 1 \cdot p_{X|Y}(X=1|Y=j) + 2 \cdot p_{X|Y}(X=2|Y=j) = \dfrac{3^{j-1}e^{-3} + 8\cdot 6^{j-1}e^{-6}}{3^{j-1}e^{-3} + 4\cdot 6^{j-1}e^{-6}}$, *j ³ 0.*

12. (2-35, 8 points) $p_{Y|X}(n|k) \sim Poisson(k)$, $p_X(k) \sim geometric_1(p)$. Using law of total probability of expectations (re. Example 2.25),

$$E[Y] = \sum_{k=1}^{\infty} E[Y \mid X=k] p^{k-1}(1-p) = \sum_{k=1}^{\infty}\left(\sum_{n=0}^{\infty} n \frac{k^n e^{-k}}{n!}\right) p^{k-1}(1-p) = \sum_{k=1}^{\infty} k \cdot p^{k-1}(1-p)$$
$$= (1-p)\frac{d}{dp}\left(\sum_{k=1}^{\infty} p^k\right) = (1-p)\frac{d}{dp}\left(\frac{1}{1-p} - 1\right) = (1-p)\cdot\frac{1}{(1-p)^2} = \frac{1}{1-p}$$

$$E[XY] = \sum_{k=1}^{\infty} E[XY \mid X=k] p_X(k) = \sum_{k=1}^{\infty} E[k\cdot Y \mid X=k] p_X(k) = \sum_{k=1}^{\infty} k \cdot E[Y \mid X=k]\cdot p_X(k)$$
$$= \sum_{k=1}^{\infty} k \cdot k \cdot p_X(k) = (1-p)\sum_{k=1}^{\infty} k^2 p^{k-1}$$

But $\dfrac{d^2 p^k}{dp^2} = k(k-1)p^{k-2}$. Hence $\displaystyle\sum_{k=2}^{\infty} k(k-1)p^{k-2} = \frac{d^2}{dp^2}\sum_{k=2}^{\infty} p^k = \frac{d^2}{dp^2}\left(\frac{1}{1-p} - 1 - p\right) = \frac{2}{(1-p)^3}$,

and $\displaystyle E[XY] = (1-p)\sum_{k=1}^{\infty}[k(k-1)+k]p^{k-1} = p(1-p)\sum_{k=2}^{\infty} k(k-1)p^{k-2} + E[Y] = \frac{1+p}{(1-p)^2}$

$$E[Y^2] = \sum_{k=1}^{\infty} E[Y^2 \mid X=k] p_X(k) = \sum_{k=1}^{\infty}\left(\sum_{n=0}^{\infty} n^2 \frac{k^n e^{-k}}{n!}\right) p_X(k) = \sum_{k=1}^{\infty}(k^2+k)\cdot p^{k-1}(1-p)$$
$$= E[XY] + E[Y] = 2/(1-p)^2$$

Hence, $Var(Y) = E[Y^2] - (E[Y])^2 = \left(\frac{2}{p(1-p)^2} + \frac{1}{1-p}\right) - \left(\frac{1}{1-p}\right)^2 = \frac{2-p}{p(1-p)^2} + \frac{1}{1-p}$

13. (2-35, 4 points) Given $X \sim Bernoulli(p)$, $p_{Y|X}(n|1) \sim Poisson(\,l)$, and $p_{Y|X}(n|0) \sim Poisson(\,l/2)$.

$E[Y|X=1] = l$ since $p_{Y|X}(n|1) \sim Poisson(\,l)$. $E[Y|X=0] = l/2$ since $p_{Y|X}(n|0) \sim Poisson(\,l/2)$.

$E[Y] = E[Y|X=1]\,p + E[Y|X=0]\,(1-p) = lp + l(1-p)/2 = l/2 + lp/2$.

Using example 2.15, $E[Y^2] = E[Y^2|X=1]\,p + E[Y^2|X=0]\,(1-p) = (\,l^2 + l)p + (\,l^2/4 + l/2)(1-p)$.

$Var(Y) = E[Y^2] - (E[Y])^2 = (1+p)\,l/2 + p(1-p)\,l^2/4$.

14. (3-3, 4 points) (a) $P(X>t) = 1 - \int_{x=0}^{t} le^{-lx}dx = 1 - \left(-(e^{-lt}-1)\right) = e^{-lt}$.

(b) Note that $\{w; X(w) > t + \Delta t\} \subset \{w; X(w) > t\}$. Hence $P(X>t+\Delta t|X>t) = P(X>t+\Delta t)/P(X>t) = e^{-\lambda(t+\Delta t)}/e^{-\lambda t} = e^{-\Delta t}$.

15. (3-4, 4 points) $X_1, \ldots, X_n \sim exp(\,l)$, *i.i.d.*

(a) $P(min(X_1, \ldots, X_n) > 2) = P(\bigcap_{i=1}^{n}\{X_i > 2\}) = \prod_{i=1}^{n} P(X_i > 2) = e^{-2nl}$

(b) $P(max(X_1, \ldots, X_n) > 2) = P(\bigcup_{i=1}^{n}\{X_i > 2\}) = 1 - P(\bigcap_{i=1}^{n}\{X_i \le 2\}) = 1 - (1 - e^{-2l})^n$

16. (3-5, 6 points) $X \sim exp(\,l)$, $Y \sim exp(m)$. X, Y are independent r.v.s.

(a) $P(Y \pounds 2) = 1 - e^{-2m}$.

(b) $P(X \pounds 12, Y \pounds 12) = P(X \pounds 12)P(Y \pounds 12) = (1 - e^{-12l})(1 - e^{-12m})$

(c) $P(X \pounds 12, \text{ or } Y \pounds 12) = P(X \pounds 12) + P(Y \pounds 12) - P(X \pounds 12, Y \pounds 12) = (1 - e^{-12l}) + (1 - e^{-12m}) - (1 - e^{-12l})(1 - e^{-12m}) = 1 - e^{-12l}e^{-12m}$.

17. (3-6, 6 points)

(a) $\frac{d}{dx}\exp(-lx^p) = -lpx^{p-1}\exp(-lx^p)$. Hence, $\int_{x=0+}^{\infty} lpx^{p-1}e^{-lx^p}dx = -\int_{x=0+}^{\infty} d\exp(-lx^p) = 1$.

(b) $P(X>t) = \int_{x=t}^{\infty} lpx^{p-1}e^{-lx^p}dx = -\int_{x=t}^{\infty} d\exp(-lx^p) = \exp(-lt^p)$, $t > 0$.

(c) $P(X_1 \pounds 3, \ldots, X_n \pounds 3) = (1 - \exp(-\,l3^p))^n$ since $X_1, \ldots, X_n \sim Weibull(p, l)$ and are iid.
$P(X_1 > 3, \text{ or } X_2 > 3, \ldots, \text{ or } X_n > 3) = 1 - P(X_1 \pounds 3, \ldots, X_n \pounds 3) = 1 - (1 - \exp(-\,l3^p))^n$

18. (3-10(a), 4 points)

(a) $\frac{d}{dx}\left(x^{p-1}e^{-x}\right) = (p-1)x^{p-2}e^{-x} - x^{p-1}e^{-x}$. Hence,

$$\int_{x=0}^{\infty} d\left(x^{p-1}e^{-x}\right) = \int_{x=0}^{\infty}(p-1)x^{p-2}e^{-x}dx - \int_{x=0}^{\infty}x^{p-1}e^{-x}dx = 0 = (p-1)\Gamma(p-1) - \Gamma(p)$$

or $\Gamma(p) = (p-1)\Gamma(p-1)$

19. (3-11(a)(b), 4 points)

(a) First, $f_l(x) = lf(lx)$ ³ 0 if $f(x)$ ³ 0 for all x, and $l > 0$. Moreover,

$$\int_{x=-\infty}^{\infty} f_l(x)dx = \int_{x=-\infty}^{\infty} lf(lx)dx = \int_{lx=-\infty}^{\infty} f(lx)d(lx) = \int_{y=-\infty}^{\infty} f(y)dy = 1.$$ Hence $f_l(x)$ is also a pdf.

(b) $g_{1,l}(x) = le^{-lx}$, $x > 0$, ~ exp(l).

20. (3-19, 4 points) $E[X] = \int_{x=0}^{\infty} x \cdot l \exp(-lx)dx = -l \cdot \frac{d}{dl}\left[\int_{x=0}^{\infty} \exp(-lx)dx\right] = -l \cdot \frac{d}{dl}\left(\frac{1}{l}\right) = \frac{1}{l}$

21. (3-20, 4 points) X_i ~ exp(l), 1 £ i £ n, iid, $E[X_i]$ = m. = 1/ l. Hence, $l = 1/m$.

(a) P(at least one $X_i < m/2$) = 1- P(every X_i ³ $m/2$) = 1- $(\exp(- l(m/2)))^n$ = 1- $(e^{-n/2})$.

22. (3-21, 4 points) $E[X] = \int_{x=1}^{\infty} x \cdot \frac{2}{x^3} dx = -2 \cdot \int_{x=1}^{\infty} d(x^{-1}) = 2$