Department of Electrical and Computer Engineering
University of Wisconsin - Madison

ECE 331 Introduction to Random Signal Analysis and Statistics

Final Examination Solution

Thursday, May 17, 2001, 7:45-9:45 AM

1. (10 points) You are to test six memory chips. Out of these 6 chips, you are told that two of them are defective. However, before you start testing, your instructor took away two chips to test.
   (a) (3 points) If your instructor tells you that both chips he tested are good. What is the probability that you test one of the remaining four chips and find it defective?
   (b) (7 points) What is the probability that one of those two chips your instructor tested is defective AND that you test one of the remaining four chips and find it defective?

   Answer: (a) 1/2. (b) P(A & B) = P(A)P(B|A) = [(2/6)(4/5)+(4/6)(2/5)] (1/4) = 2/15 = 0.1333

2. (25 points)
   (a) (7 points) Let X and Y be i.i.d. random variables such that $E[X] = m_X$, $E[Y] = m_Y$, $Var(X) = \sigma_X^2$, $Var(Y) = \sigma_Y^2$. Define Z = (X + Y)/2. Find E[Z] (2 points) and Var(Z) (5 points)

   Answer: $E[Z] = (E[X] + E[Y])/2 = (m_X + m_Y)/2$. $Var(Z) = E[(Z - E[Z])^2] = (1/4)E[(X-m_X) + (Y-m_Y)]^2 = (1/4) \{E[(X-m_X)^2] + E[(Y-m_Y)^2] + E[(X-m_X) (Y-m_Y)] \} = (\sigma_X^2+\sigma_Y^2)/4$. Note that $E[(X-m_X) (Y-m_Y)] = E[(X-m_X)] E[(Y-m_Y)] = 0$.

   (b) (6 points) If X and Y are independent Bernoulli(p) random variables, find E[Y/(X+1)].

   Answer: E[Y/(X+1)] = E[Y]E[1/(X+1)] = p {1 (1–p) + (1/2) p} = p(1–p/2).

   (c) (6 points) Let $X_1, X_2, \ldots, X_n$ be a set of i.i.d. uniform[0, 1] random variables. Find $P(Max(X_1, X_2, \ldots, X_n) > 0.8)$.

   Answer: $P(X_i \le 0.8) = 0.8$. $P(Max(X_1, X_2, \ldots, X_n) > 0.8) = 1-P(Max(X_1, X_2, \ldots, X_n) \le 0.8) = 1 - P(X_1 \le 0.8, \ldots, X_n \le 0.8) = 1 - (0.8)^n$.

   (d) (6 points) In a casino game, three fair dices are tossed simultaneously. A player will bet on an integer number between 1 and 6. If that number matches the outcome of ANY of the three dices, the player wins. What is the probability that a player wins the bet?

   Answer: $P(lose) = (5/6)^3 = 125/216$. Hence P(win) = 1 – P(lose) = 1 – 125/216 = 91/216 = 0.4213.

3. (15 points)
   (a) (5 points) Let X ~ uniform[–2,1], and $F_X(x)$ be its c.d.f. Find $F_X(2) - F_X(-1)$.

   Answer: $F_X(2) = 1$, $F_X(-1) = 1/3$. Hence $F_X(2) - F_X(-1) = 2/3$.

   (b) (10 points) Let X ~ uniform[0,1]. If Y = g(X) = 2X + 1. Find the the cdf $F_Y(y)$ and the pdf $f_Y(y)$.

   Answer: $B = \{X;\ 2X + 1 \le y\} = \{X \le (y-1)/2\}$. $F_Y(y) = F_X((y-1)/2) = 0$ if $y < 1$, $= (y-1)/2$ for $1 \le y < 3$, $= 1$ for $3 \le y$. Hence $f_Y(y) = 0.5I_{[1,3]}$. That is, Y ~ uniform[1,3].

4. (20 points)
   (a) (5 points) Determine if $S_Y(f) = (e^{j2f} + e^{-j2f} + 2)/2$ is a valid power spectral density function. Explain your answer briefly. (No explanation, no credit).

   Answer: Yes. Using Euler's formula, $S_Y(f) = 1 + \cos(2f)$ is an real, even, and non-negative function of f.

   (b) (5 points) Can $R(\tau) = I_{[-1,1]}(\tau)$ be a valid auto-correlation function of a WSS random process? (*Hint: Use Fourier transform table provided*).

   Answer: No. Because the Fourier transform of $I_{[-1,1]}(\tau)$ is $2Sin(2pf)/(2pf)$ which may assume negative values for certain frequency, and hence can not be a valid power spectrum. Therefore, $R(\tau)$ is not a valid auto-correlation function.

   (c) (10 points) Given that X_t is a WSS random process with auto-correlation function $R(\tau) = 3e^{-2|t|}$. Find the power spectral density function $S_X(f)$ (7 points) and the total power P_X (3 points).

   Answer: $S_X(f) = 12/(4 + 4\pi^2 f^2) = 3/(1+\pi^2 f^2)$ by Fourier transformation table, and $P_X = R(0) = 3$.

5. (10 points) Let N_t be a Poisson process with rate $\lambda = 1$, and consider a fixed observation interval (0, 6]. What is the probability that $N_i - N_{i-1} = 2$ for every time slots $1 \le i \le 5$? You can leave the natural exponent *e* in the answer.

Answer: Since $N_i - N_{i-1} \sim \text{Poisson}(\lambda)$, $P(N_i - N_{i-1} = 2) = \lambda^2 e^{-\lambda}/2! = e^{-1}/2$. Also, each random variables of $N_i - N_{i-1}$ for different i are independent. Hence,

$$P\left(\bigcup_{i=1}^{5}\{N_i - N_{i-1} = 2\}\right) = \left(P(N_i - N_{i-1} = 2)\right)^5 = \left(\frac{l^2 e^{-l}}{2!}\right)^5 = \frac{1}{32e^5} = 0.00021$$

6. (10 points) Let X_i be i.i.d., uniform[−1,1] random variables with zero mean ($E[X_i] = 0$) and variance $Var(X_i) = \sigma^2 = 1/3$. Define $S_n = \sum_{i=1}^{n} X_i$. Use central limit theorem, find $P(S_3 > 2)$.

   Express the answer in terms of the CDF of standard normal distribution denoted by

   $$\Phi(y) = \frac{1}{\sqrt{2p}} \int_{-\infty}^{y} e^{-x^2/2} dx.$$

Answer: $P(S_3 > 2) = P(M_3 > 2/3) = P((M_3 - 0)/(\sigma/\sqrt{(3)}) > (2/3-0)/(1/3)) = P(Y_3 > 2) =$ **1** - F **(2) = 0.0047.**

7. (10 points) Y_i is i.i.d ~ Bernoulli(0.5). Define $M_n = \frac{1}{n}\sum_{i=1}^{n} X_i$. If $M_{100} = 0.37$, find the 99% confidence interval of M_{100}.

Answer: Note that $\sigma^2 = 0.5(1-0.5) = 0.25$. Since $1 - \alpha = 0.99$, based on table 1 given at the end of the exam, we have $y_{a/2} = 2.576$. The confidence interval therefore is $[M_{100} - \sigma y_{a/2}/\sqrt{(100)}, 0.37 + (0.5)(2.576)/10] = [0.2412, 0.4988]$.