Department of Electrical and Computer Engineering
University of Wisconsin - Madison

ECE 331 Introduction to Random Signal Analysis and Statistics

Examination #2 Solution

Friday, March 9, 2001, 12:05 – 12:55 PM

1. (25 points) *Expectation, moment*

   (a) (5 points) Let Y be a random variable. If $E[Y] = a$, and $E[Y^2] = b$. Find $Var(Y)$.

   Answer: $Var[Y] = \mathbf{b} - \mathbf{a^2}$.

   (b) (10 points) A discrete random variable X has the following *pmf*

k	1	2	3	4	Other values
$p_X(k)$	0.1	0.3	0.4	0.2	0

   Find the corresponding probability generation function (pgf) $G_X(z)$.

   Answer: $G_X(z) = E[z^X] = \mathbf{0.1z+0.3z^2+0.4z^3+0.2z^4}$

   (c) (10 points) For the same random variable X defined in part (b), find $E[1/(X+1)]$

   Answer: $E[1/(X+1)]$ = 0.1*(1/(1+1))+ 0.3*(1/(2+1)) + 0.4*(1/(3+1)) + 0.2*(1/(4+1)) = 0.1*(1/2) + 0.3*(1/3)+ 0.4*(1/4) + 0.2*(1/5) = 0.05 + 0.1 + 0.1 + 0.04 = **0.29**

2. (20 points) *Conditional probability*

   Let X and Y be two i.i.d. random variables with

$$p_X(k) = p_Y(k) = \begin{cases} 0.2 & k=0 \\ 0.3 & k=1 \\ 0.5 & k=2 \\ 0 & otherwise. \end{cases}$$

   If $Z = X+Y$. Find $p_Z(j)$. *Hint: Example 2.21*

   Answer:

$$p_Z(j) = \sum_{i=\max(0,j-2)}^{\min(j,2)} p_Y(j-i)p_X(i) = \begin{cases} 0.2*0.2 & j=0 \\ 0.2*0.3+0.3*0.2 & j=1 \\ 0.2*0.5+0.3*0.3+0.5*0.2 & j=2 \\ 0.3*0.5+0.5*0.3 & j=3 \\ 0.5*0.5 & j=4 \\ 0 & otherwise. \end{cases} = \begin{cases} 0.04 & j=0 \\ 0.12 & j=1 \\ 0.29 & j=2 \\ 0.3 & j=3 \\ 0.25 & j=4 \\ 0 & otherwise. \end{cases}$$

3. (30 points) *Conditional expectation, total probability, substitution law*

   Let $p_{Y|X}(n|k) = P(Y=n|X=k) = k^n e^{-k}/n!$, $n = 0, 1, 2, \ldots$, and X be a discrete random variable with *pmf* $p_X(1) = 0.25$, $p_X(2) = 0.5$, *and* $p_X(3) = 0.25$.

   (a) (15 points) Find $E[Y^{X/2}|X=2]$.

Answer: With the substitution law, $\boldsymbol{E[Y^{X/2}|X=2] = E[Y^{2/2}|X=2] = E[Y|X=2] = 2}$**.** The last equality is due to the observation that $p_{Y|X}(n|2)$ has a Poisson(2) pmf and the expectation of a Poisson(2) random variable is 2. Note that the information about X is not used except to validate that $P(X=2) > 0$.

(b) (15 points) Find $E[Y]$

Answer:

E[Y] = E[Y|X=1]p_X(1) + E[Y|X=2]p_X(2) + E[Y|X=3]p_X(3) =1*0.25 + 2*0.5 + 3*0.25 **= 2**.

4. (25 points) *Continuous random variable, moments*

Let X be a continuous random variable with a *pdf* ~ uniform[a, b].

(a) (10 points) Find its moment generating function $M_X(s)$.

Answer: $M_X(s) = E[e^{sX}] = \int_{x=-\infty}^{\infty} e^{sx} f_X(x)dx = \int_{x=a}^{b} e^{sx}\frac{1}{b-a}dx = \frac{e^{sb}-e^{sa}}{(b-a)s}$

(b) (15 points) Given that $E[X] = (b+a)/2$, derive the variance Var(X).

Answer:

$$Var(X) = E[(X-E[X])^2] = \frac{1}{b-a}\int_{x=a}^{b}\left(x-\frac{b+a}{2}\right)^2 dx = \frac{1}{b-a}\int_{u=-\frac{b-a}{2}}^{\frac{b-a}{2}} u^2 du$$

$$= \frac{1}{b-a}\cdot\frac{u^3}{3}\Bigg|_{-(b-a)/2}^{(b-a)/2} = \frac{(b-a)^2}{12}$$