

ECE334 HOMEWORK 2: SOLUTIONS TO QUESTIONS 1,2,3

1 [linear independence and bases] Prove that in a vector space of dimension n , every set of n linearly independent vectors is a basis.

Let the set of n linearly independent vectors be $\{v_1, v_2, \dots, v_n\}$. Let x be any vector. Then $\{x, v_1, v_2, \dots, v_n\}$ is a set of $n+1$ vectors in a vector space of dimension n . The dimension n is defined to be the maximum number of linearly dependent vectors. Therefore $\{x, v_1, v_2, \dots, v_n\}$ is linearly dependent. That is, there are scalars $\lambda_0, \lambda_1, \dots, \lambda_n$, not all zero, such that

$$\lambda_0 x + \lambda_1 v_1 + \lambda_2 v_2 + \dots + \lambda_n v_n = 0$$

It is impossible that $\lambda_0 = 0$ since this would imply that $\{v_1, v_2, \dots, v_n\}$ are linearly dependent which contradicts our initial assumption. Therefore we can divide by λ_0 and conclude that x is a linear combination of $\{v_1, v_2, \dots, v_n\}$:

$$x = -(\lambda_1/\lambda_0)v_1 - (\lambda_2/\lambda_0)v_2 - \dots - (\lambda_n/\lambda_0)v_n$$

or, rewriting, there are scalars $\mu_1, \mu_2, \dots, \mu_n$ with

$$x = \mu_1 v_1 + \mu_2 v_2 + \dots + \mu_n v_n \tag{1}$$

We have shown that any vector x is a linear combination of $\{v_1, v_2, \dots, v_n\}$. To finish proving that $\{v_1, v_2, \dots, v_n\}$ is a basis we also need to show that the linear combination is unique. Suppose that x is also given by the linear combination

$$x = \mu'_1 v_1 + \mu'_2 v_2 + \dots + \mu'_n v_n \tag{2}$$

Subtracting (2) from (1) gives

$$0 = (\mu_1 - \mu'_1)v_1 + (\mu_2 - \mu'_2)v_2 + \dots + (\mu_n - \mu'_n)v_n$$

Since $\{v_1, v_2, \dots, v_n\}$ are linearly independent, this implies that $\mu_j - \mu'_j = 0$ and $\mu_j = \mu'_j$ for $j = 1, \dots, n$. We have proved that $\{v_1, v_2, \dots, v_n\}$ is a basis.

2 [change of basis] Let α be a linear map $\mathbb{R}^3 \rightarrow \mathbb{R}^3$ and $\{e_1, e_2, e_3\}, \{f_1, f_2, f_3\}$ be bases for \mathbb{R}^3 . $\{f_1, f_2, f_3\}$ is related to $\{e_1, e_2, e_3\}$ by $f_j = \sum_k P_{jk} e_k$, $j = 1, 2, 3$ where P is an invertible matrix. The matrix A of α with respect to the basis $\{e_1, e_2, e_3\}$ is given by $\alpha(e_j) = \sum_i A_{ij} e_i$, $j = 1, 2, 3$. Derive the formula for the matrix of α with respect to the basis $\{f_1, f_2, f_3\}$.

To obtain the matrix \tilde{A} of α with respect to the basis $\{f_1, f_2, f_3\}$ we need to work out $\alpha(f_j)$ as a linear combination of $\{f_1, f_2, f_3\}$. (\tilde{A} satisfies $\alpha(f_j) = \sum_\ell \tilde{A}_{j\ell} f_\ell$.)

$$\alpha(f_j) = \alpha\left(\sum_k P_{jk} e_k\right) = \sum_k P_{jk} \alpha(e_k) = \sum_k P_{jk} \sum_i A_{ik} e_i \tag{1}$$

Now we need to express $\{e_1, e_2, e_3\}$ in terms of $\{f_1, f_2, f_3\}$. Suppose that $e_i = \sum_k Q_{ik} f_k$. We claim that $Q = P^{-1}$.

$$e_i = \sum_\ell Q_{i\ell} f_\ell = \sum_\ell Q_{i\ell} \sum_j P_{\ell j} e_j = \sum_j \sum_\ell Q_{i\ell} P_{\ell j} e_j$$

so that $\sum_\ell Q_{i\ell} P_{\ell j} = \delta_{ij}$ and $Q = P^{-1}$. Equation (1) becomes

$$\begin{aligned} \alpha(f_j) &= \sum_k P_{jk} \sum_i A_{ik} \sum_\ell Q_{i\ell} f_\ell \\ &= \sum_\ell \left(\sum_{i,k} P_{jk} A_{ik} Q_{i\ell} \right) f_\ell \\ &= \sum_\ell \left(\sum_{i,k} Q_{\ell i}^T A_{ik} P_{kj}^T \right) f_\ell \end{aligned}$$

Therefore $\tilde{A} = \sum_{i,k} Q_{\ell i}^T A_{ik} P_{kj}^T$, or, in matrix notation,

$$\tilde{A} = Q^T A P^T = P^{T^{-1}} A P^T$$

3 [direct sum and bases] Let V and W be vector spaces with corresponding bases $\{v_1, v_2, \dots, v_j\}$ and $\{w_1, w_2, \dots, w_k\}$. Prove that $\{v_1, v_2, \dots, v_j, w_1, w_2, \dots, w_k\}$ is a basis of $V \oplus W$ and deduce that $\text{dimension}(V \oplus W) = \text{dimension}(V) + \text{dimension}(W)$. Give an example to show that it is not generally true that $\text{dimension}(V + W) = \text{dimension}(V) + \text{dimension}(W)$.
 Let x be any vector in $V \oplus W$. Then, by definition of $V + W$, $x = v + w$ for v in V and w in W . v in V implies that $v = \lambda_1 v_1 + \lambda_2 v_2 + \dots + \lambda_j v_j$ and w in W implies that $w = \mu_1 w_1 + \mu_2 w_2 + \dots + \mu_k w_k$. Therefore

$$x = \lambda_1 v_1 + \lambda_2 v_2 + \dots + \lambda_j v_j + \mu_1 w_1 + \mu_2 w_2 + \dots + \mu_k w_k \quad (1)$$

Thus any vector x is a linear combination of $v_1, v_2, \dots, v_j, w_1, w_2, \dots, w_k$.

To finish showing that $\{v_1, v_2, \dots, v_j, w_1, w_2, \dots, w_k\}$ is a basis, we need to show that linear combination (1) is unique. Suppose also that

$$x = \lambda'_1 v_1 + \lambda'_2 v_2 + \dots + \lambda'_j v_j + \mu'_1 w_1 + \mu'_2 w_2 + \dots + \mu'_k w_k \quad (2)$$

Subtract (2) from (1) and rearrange to obtain

$$(\lambda_1 - \lambda'_1)v_1 + (\lambda_2 - \lambda'_2)v_2 + \dots + (\lambda_j - \lambda'_j)v_j = (\mu'_1 - \mu_1)w_1 + (\mu'_2 - \mu_2)w_2 + \dots + (\mu'_k - \mu_k)w_k \quad (3)$$

The left hand side of (3) is in V and the right hand side of (3) is in W , so both sides are in $V \cap W$. But $V \oplus W$ is a direct sum and so $V \cap W = \{0\}$ and the left and the right hand side of (3) must both be zero. Since $\{v_1, v_2, \dots, v_j\}$ is linearly independent, $(\lambda_i - \lambda'_i) = 0$ and $\lambda_i = \lambda'_i$ for $i = 1, \dots, j$. Similarly, since $\{w_1, w_2, \dots, w_k\}$ is linearly independent, $\mu_i = \mu'_i$ for $i = 1, \dots, k$. Therefore the linear combination (1) is unique.