ECE334 HOMEWORK 2

1 [linear independence and bases] Prove that in a vector space of dimension n, every set of n linearly independent vectors is a basis.

2 [change of basis] Let α be a linear map $\mathbb{R}^3 \to \mathbb{R}^3$ and $\{e_1, e_2, e_3\}$, $\{f_1, f_2, f_3\}$ be bases for $\mathbb{R}^3$. $\{f_1, f_2, f_3\}$ is related to $\{e_1, e_2, e_3\}$ by $f_j = \sum_k P_{jk} e_k$, $j = 1, 2, 3$ where P is an invertible matrix. The matrix A of α with respect to the basis $\{e_1, e_2, e_3\}$ is given by $\alpha(e_j) = \sum_i A_{ij} e_i$, $j = 1, 2, 3$. Derive the formula for the matrix of α with respect to the basis $\{f_1, f_2, f_3\}$.

3 [direct sum and bases] Let V and W be vector spaces with corresponding bases $\{v_1, v_2, \ldots v_j\}$ and $\{w_1, w_2, \ldots w_k\}$. Prove that $\{v_1, v_2, \ldots v_j, w_1, w_2, \ldots w_k\}$ is a basis of $V \oplus W$ and deduce that dimension($V \oplus W$)=dimension(V)+dimension(W). Give an example to show that it is not generally true that dimension($V + W$)=dimension(V)+dimension(W).

4 [kernels and ranges] Let α be a linear map from $\mathbb{R}^3 \to \mathbb{R}^2$. Write down the definitions of kerα and rangeα. Prove that kerα is a subspace of $\mathbb{R}^3$ and that rangeα is a subspace of $\mathbb{R}^2$. Give an example of α with a 2 dimensional kernel and a one dimensional range.

5 [a real life example of kernels and ranges] Let

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 2 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

What is the rank of A? Find a basis of the kernel K and range R of A. Draw pictures of K and R in $\mathbb{R}^3$. Under what conditions does $Ax = b$ have a solution for x and is the solution unique?