ECE 901
Homework 9

Many problems in electrical engineering and communications involve the recovery of sinusoidal signals from noisy measurements. Suppose we make observations of the following form

$$Y_i \;=\; \sum_{k=1}^{K} A_k \, e^{j2\pi\omega_k(i-1)} \;+\; W_i \;, \quad i = 1, \ldots, n$$

where $j = \sqrt{-1}$, A_k are complex numbers, $\omega_k \in [0, 2\pi]$, $K < n$, and $W_i \overset{iid}{\sim} \mathcal{N}(0, \sigma^2)$. In words, we observe the sum of K complex sinusoids in Gaussian white noise. In vector notation, we have

$$Y \;=\; \sum_{k=1}^{K} A_k \, \phi(\omega_k) \;+\; W$$

where $\phi(\omega_k) = [1,\, e^{j2\pi\omega_k},\, \ldots,\, e^{j2\pi\omega_k(n-1)}]^T$. The goal is to estimate the sinusoidal signal

$$f^* = \sum_{k=1}^{K} A_k \, \phi(\omega_k)$$

given the noisy observations Y.

1. Suppose that K and the frequencies $\omega_1, \ldots, \omega_K$ are known. Propose an estimator for f^* in this situation. Derive an explicit expression for the MSE (Hint: standard multivariate statistical analysis will suffice here).

2. In many problems of interest we do not know the frequencies, but we may have reason to believe that the signal is of the form of f^* for some unknown value of K and the parameters $\{(A_k, \omega_k)\}_{k=1}^{K}$. Propose an estimator for f^* in this case and derive an upper bound for its MSE. Show that the MSE is nearly the same as in the case when K and the frequencies are known.

3. Propose an efficient algorithm for computing the estimator in 2 above. It is possible to devise an algorithm whose computational complexity is $O(n \log n)$. Simulate the estimator in Matlab and experiment with different numbers of sinusoids and noise levels. Bear in mind that the unknown frequencies could take any value in $[0, 2\pi]$.